DOCUMENT RESUME

ED 033 847 SE 007 211

AUTHOR	Friedman, Bernard
TITLE	[Geometry Through Symmetry, Cambridge Conference on School Mathematics Feasibility Study No. 32.]
INSTITUTION	Cambridge Conference on School Mathematics, Newton, Mass.
Pub Date	[69]
Note	58p.
EDRS Price	EDRS Price MF-$0.50 HC Not Available from EDRS.
Descriptors	Elementary School Mathematics, *Geometric Concepts, *Geometry, Grade 8, *Instruction, *Instructional Materials, *Mathematics, Resource Materials, Secondary School Mathematics
Identifiers	Cambridge Conference on School Mathematics

Abstract

These materials were written for the use of a class of eighth grade high ability students in a four week course sponsored by Educational Services Incorporated on the Stanford campus. They represent a practical response to the proposal by the Cambridge Conference of 1963 that geometry be taught by vector space methods. Instead of using vector methods, these materials represent an attempt to obtain the geometrical properties of figures from proofs and arguments about their symmetry properties. These notes contain instructional materials on such mathematical concepts as reflection in the plane, perpendicularity, central symmetry, translation of the plane, and rotation. In addition, these notes contain definitions, exercises, and summaries of results obtained in class. [Not available in hardcopy due to marginal legibility of original document.] (RP)

ED033847

CCSM
Feasibility Study No. 32

Cambridge Conference on School Mathematics - 1964
Stanford, California

Bernard Friedman
Eighth Grade

The following set of notes was written for the use of a class of eighth-grade students in a four week course sponsored by Educational Services Incorporated on the Stanford campus. The background for this course is as follows:

The Cambridge Conference of 1963 on Goals in School Mathematics proposed that geometry be taught by vector space methods. At theMiami, 1964, meeting of the participants of the Conference, I volunteered to experiment with such a course during the summer of 1964. With the help of Professor Pat Suppes, it was arranged that I would work with a class of eighth-grade children to whom he had been teaching Mathematical Logic for the past two years. Since the children had not had any training in algebra, I doubted the possibility of doing anything worthwhile with vector methods. Incidentally, the children's lack of algebraic skills became apparent during the last session of the course when I tried to present the standard fallacious argument that 1=0 after division by zero. Even though the students had had during the previous summer a four weeks session on the field properties of the real number system, they were uncertain about the simple algebraic manipulations necessary for the argument.

Instead of using vector methods, I tried to obtain the geometrical properties of figures from arguments about their symmetry properties. No attempt was made to introduce a formal axiom system. The class was led to learn that diagrams and intuition may be misleading, so that some sort of proof was necessary. The proof was based on certain explicit and implicit assumptions. The explicit assumptions were roughly the following:

1. The plane is a metric space
2. A line is a set of points such that the triangle inequality is satisfied for every three points in the set
3. Given two points A and B and two real numbers and there are either zero, one, or two points P such that the distance from P to A equals and the distance from P to B, . Whether the number is zero, one or two depends on the triangle inequality.
4. To every line corresponds a distance-preserving mapping (reflection) of the plane into itself which leaves the line fixed.

SE007 211

The implicit assumptions were the customary ones about incidence, order, etc. in the Euclidean plane.

The class met for an hour every day for four weeks, from June 22 to July 17, 1964. Homework was assigned daily, collected, and returned with corrections. The class started with twenty-one students. The number remained constant for three weeks, but during the last week it dropped to eighteen.

At the beginning of the course, I assumed that the students would be almost completely ignorant of geometry. However, after they tossed around such concepts as "perpendicular bisector" and "collinearity" during the first few lessons, I decided to work with them on a more sophisticated level. The notes reflect this change of level because they begin at what I assumed to be an eighth-grade level and soon jumped to a level suitable for college freshmen.

The notes contain definitions, exercises, and summaries of results obtained in class. All of the exercises through page 32 were done by the students. A few of the later exercises were discussed in class in the last session. We just reached the point where a group could be defined.

The students were not tested at any time, but the homework was carefully checked and their reactions during the class sessions constantly observed. The following indications of learning may be noted:

1. The ability to present a proof of a conjecture increased until at the end most of the students could give a reasonable argument for the validity of their statements.
2. One of the students, in response to the question "Does a parallelogram have an axis of symmetry?" pointed out that a line through theintersection of the diagonals perpendicular to the plane of the parallelogram was an axis of symmetry. This answer shows a surprising insight because we had restricted our discussion to axes in a plane or on the surface of a sphere.
3. One of the students discovered and "proved" that the composition of two symmetries of a figure is also a symmetry of the figure.

Because of the high I.Q. of the students (the range was from 130-180), the results obtained may not be valid for average or slightly above average eighth graders. However, I believe that this experiment shows that the approach to geometry through transformations can be interesting to young students and that it can lead naturally to a desire for an axiomatic formulation.

14 September 1964

5050-64

Foreword

The following set of notes was written for the use of a class of eighth-grade students in a four week course sponsored by Educational Services Incorporated on the Stanford campus. The background for this course is as follows:

The Cambridge Conference of 1963 on Goals in School Mathematics proposed that geometry be taught by vector space methods. At the Miami, 1964, meeting of the participants of the Conference, I volunteered to experiment with such a course during the summer of 1964. With the help of Professor Pat Suppes, it was arranged that I would work with a class of eighth-grade children to whom he had been teaching Mathematical Logic for the past two years. Since the children had not had any training in algebra, I doubted the possibility of doing anything worthwhile with vector methods. Incidentally, the children's lack of algebraic skills became apparent during the last session of the course when I tried to present the standard fallacious argument that 1=0 after division by zero. Even though the students had had during the previous summer a four weeks session on the field properties of the real number system, they were uncertain about the simple algebraic manipulations necessary for the argument.

Instead of using vector methods, I tried to obtain the geometrical properties of figures from arguments about their symmetry properties. No attempt was made to introduce a formal axiom system. The class was led to learn that diagrams and intuition may be misleading, so that some sort of proof was necessary. The proof was based on certain explicit and implicit assumptions. The explicit assumptions were roughly the following:

1. The plane is a metric space
2. A line is a set of points such that the triangle inequality is satisfied for every three points in the set
3. Given two points A and B and two real numbers α and β there are either zero, one, or two points P such that the distance from P to A equals α and the distance from P to B, β. Whether the number is zero, one or two depends on the triangle inequality
4. To every line corresponds a distance-preserving mapping (reflection) of the plane into itself which leaves the line fixed.

The implicit assumptions were the customary ones about incidence, order, etc. in the Euclidean plane.

The class met for an hour every day for four weeks, from June 22 to July 17, 1964. Homework was assigned daily, collected, and returned with corrections. The class started with twenty-one students. The number remained constant for three weeks, but during the last week it dropped to eighteen.

At the beginning of the course, I assumed that the students would be almost completely ignorant of geometry. However, after they tossed around such concepts as "perpendicular bisector" and "collinearity" during the first few lessons, I decided to work with them on a more sophisticated level. The notes reflect this change of level because they begin at what I assumed to be an eighth-grade level and soon jumped to a level suitable for college freshmen.

The notes contain definitions, exercises, and summaries of results obtained in class. All of the exercises through page 32 were done by the students. A few of the later exercises were discussed in class in the last session. We just reached the point where a group could be defined.

The students were not tested at any time, but the homework was carefully checked and their reactions during the class sessions constantly observed. The following indications of learning may be noted:

1. The ability to present a proof of a conjecture increased until at the end most of the students could give a reasonable argument for the validity of their statements.

2. One of the students, in response to the question "Does a parallelogram have an axis of symmetry?" pointed out that a line through the intersection of the diagonals perpendicular to the plane of the parallelogram was an axis of symmetry. This answer shows a surprising insight because we had restricted our discussion to axes in a plane or on the surface of a sphere.

3. One of the students discovered and "proved" that the

composition of two symmetries of a figure is also a symmetry of the figure.

Because of the high I. Q. of the students (the range was from 130 -- 180), the results obtained may not be valid for average or slightly above average eighth graders. However, I believe that this experiment shows that the approach to geometry through transformations can be interesting to young students and that it can lead naturally to a desire for an axiomatic formulation.

If a figure can be folded along a line in such a way that one part of the figure fits exactly onto another part with nothing left over and nothing omitted, then the line of the fold is an <u>axis of symmetry</u> of the figure. For example, the dotted lines are axes of symmetry for the following figures:

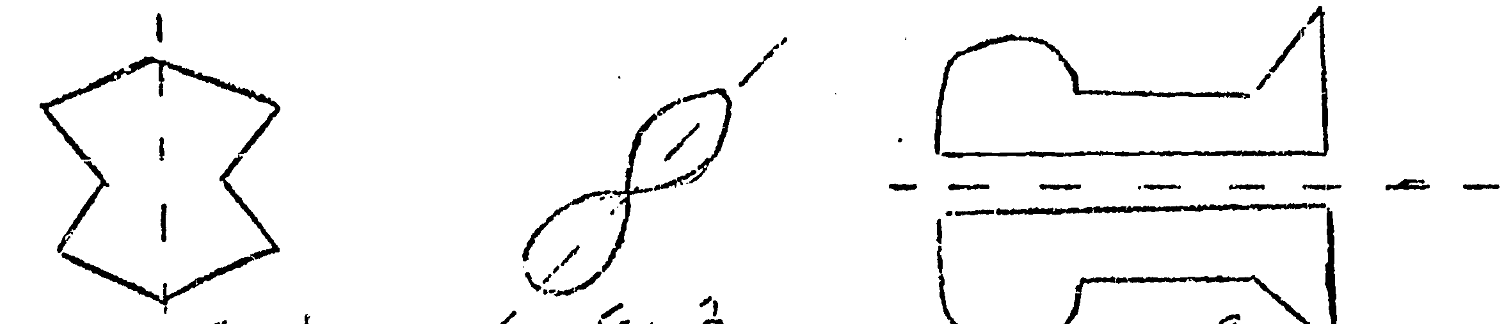
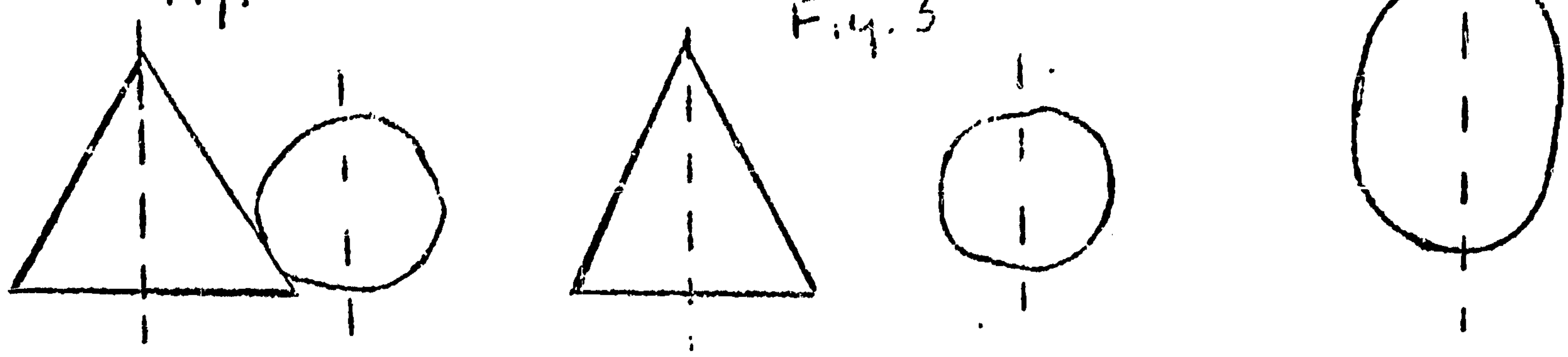


Fig. 1 Fig. 2 Fig. 3

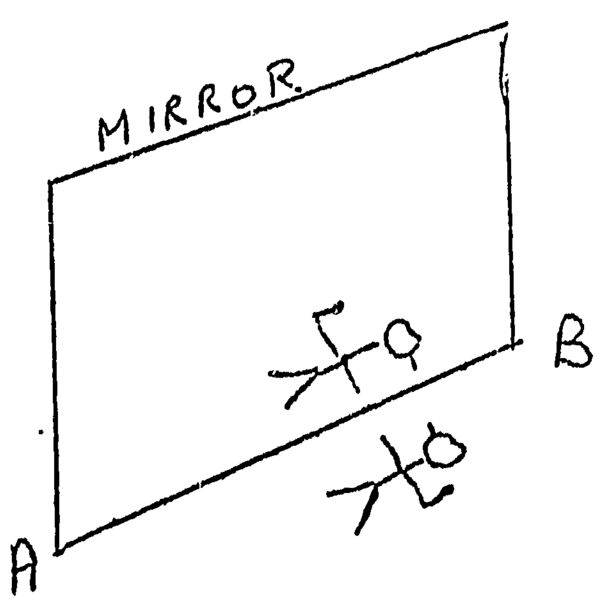
In the following examples, the dotted lines are <u>not</u> axes of symmetry for the complete figures:

Fig. 4 Fig. 5 Fig. 6



A figure which has an axis of symmetry is said to be <u>symmetric</u> with respect to that axis.

When a figure is looked at in a mirror, the original figure together with its image in the mirror form a combined figure which has the edge of the mirror as an axis of symmetry. For example in Figure 7 the edge AB is an axis of symmetry for the man and his image.



Given a part of a symmetric figure and the axis of symmetry, we can imagine a mirror and reflect in the axis the given part of the figure to get the whole figure. For example, in Figure 8 we begin with the solid part of the figure and by reflection in the axis we get the dotted part.

Fig 8

We can also imagine reflection on both sides of the axis as in Figure 9.

Fig. 9

We shall call the solid portion the original and the dotted part the image.

Exercises

In these exercises "draw" means to sketch freehand, "construct" means to make use of a ruler and compass.

1. Draw a horizontal axis of symmetry for Figure 1.

2. Draw another axis of symmetry for Figure 2.

3. Here are the letters of the English alphabet:

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z.

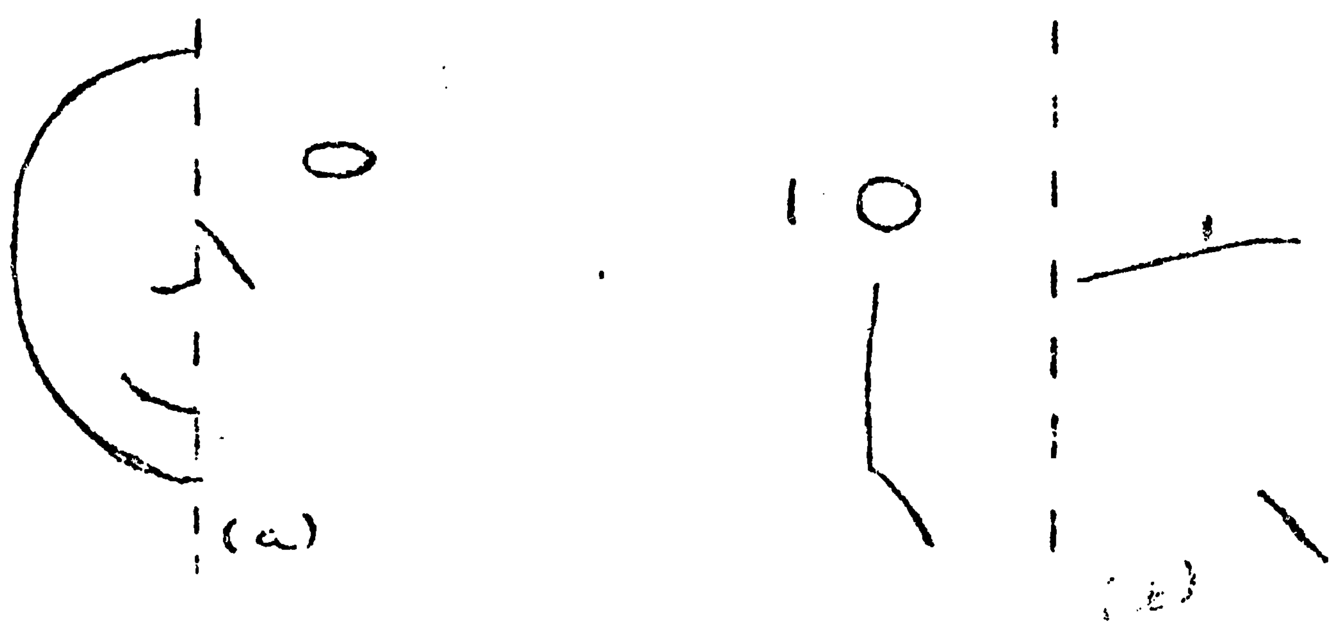
Which of these letters have a vertical axis of symmetry? Which have a horizontal axis of symmetry? Which have both axes?

4. Draw a triangle which has only one axis of symmetry. Draw a triangle which has two axes of symmetry. Does the second triangle have only two axes of symmetry? Can you find a triangle which has only two axes of symmetry? Why?

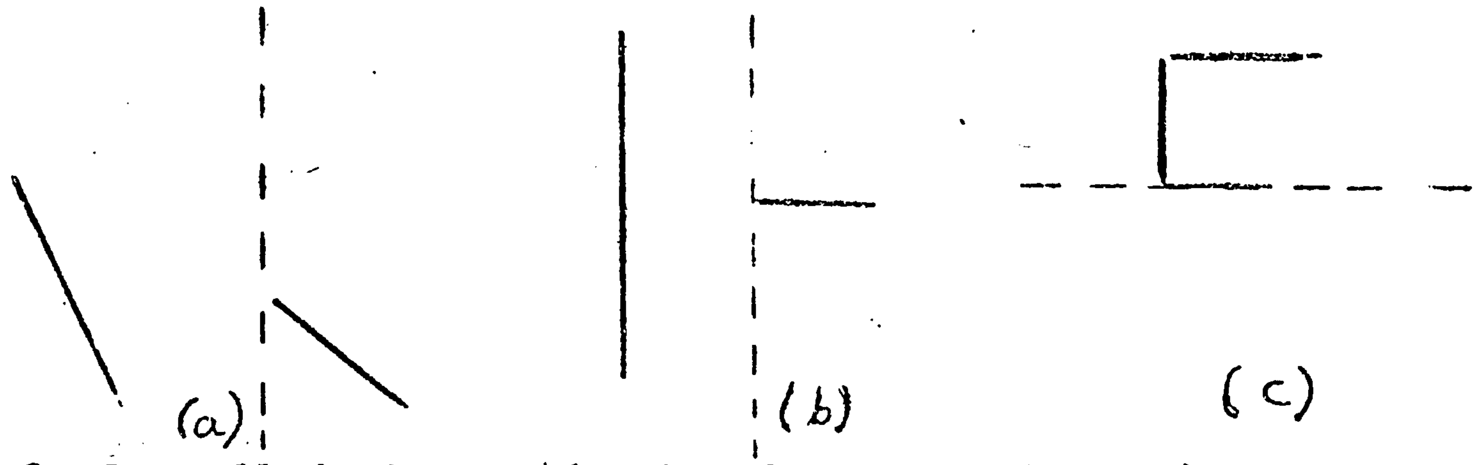
5. How many axes of symmetry does a rectangle have; does a square have? How many axes of symmetry does a circle have?

6. Draw a figure which has only five axes of symmetry. Draw one which only has six axes of symmetry.

7. Draw the image of the following original figures in the given axes:



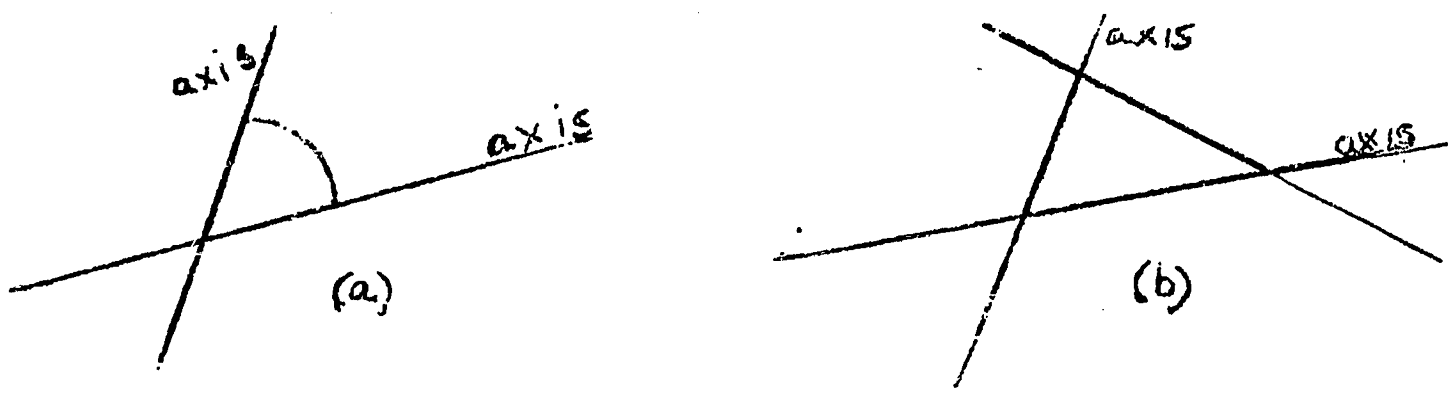
8. Without copying or completing the following diagrams, state in each case what letter of the alphabet is obtained by completing the following reflections:



9. Draw all the images (that is, also images of images) of the following in both a horizontal and a vertical axis:

Does it matter in which axis you reflect first? Why?

10. Draw all the images of the following in both of the given axes:



Does it make a difference in which axis you reflect first? Why?

A *point* is *invariant* under reflection if the point coincides with its image. A *line* is *invariant* under reflection if the line coincides with its image. A *figure* symmetric with respect to an axis is also said to be *invariant* under reflection in that axis.

Two lines are *perpendicular* if each is invariant under reflection with respect to the other as an axis. For example, Figure 10 shows two perpendicular lines.

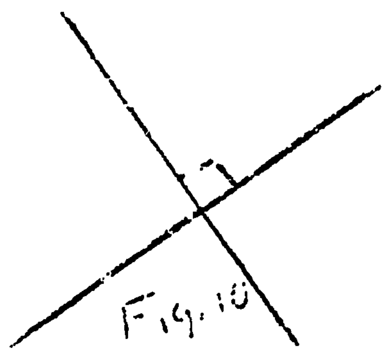
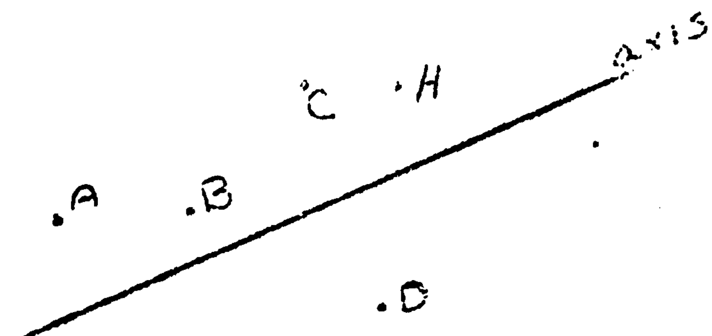


Fig. 10

The angle between two perpendicular lines is called a *right angle* and is usually marked with the symbol ⌉.

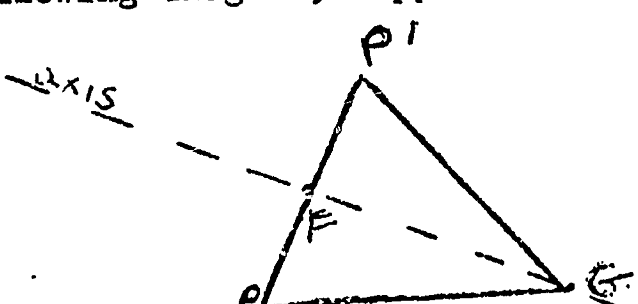
Exercises II

1. State which pairs of points in the following diagram are symmetric with respect to the axis:



2. Which points of the plane are invariant under reflection in an axis?

3. In the following diagram, suppose P' is the image of P in the axis:



What is the image of G? What is the image of P? What is the image of the line segment PG? How does the length of P'G compare to that of PG? Suppose H is any other point on the axis. How does the length of PH compare with that of P'H? Why?

Suppose the line segment PP' intersects the axis in the point F. What is the image of F? How does PF compare with P'F?

4. Fill in the blanks to form true statements.

If P is symmetric to P' with respect to an axis, then the _______ of the line segment PP' is on the axis. Also, if J is any point on the axis, the _________ between _________ and ________ is ____________ to the distance between ________ and ____________.

5. a. Suppose P and Q are equidistant from a point J on a line. Are P and Q necessarily symmetric with respect to that line as axis? Make a diagram to illustrate your answer.

b. Suppose the midpoint of the line segment joining A and B is on a given line. Are A and B necessarily symmetric with respect to that line as axis? Make a diagram to illustrate your answer.

6. Suppose J and K are two given points. What is the set of points P such that the distance from J to P is equal to α ? What is the set of points Q such that the distance from K to Q is equal to β ? When do these sets have a non-empty intersection? How many points can be in the intersection? Make diagrams illustrating your answer.

7. Suppose J and K are two given points. What is the set of points P such that JP= α and KP= β ? How many such points can there be on one side of the line segment JK?

8. Suppose J, K, P, Q are four distinct points such that JP=JQ and KP=KQ. Draw the line through J and K and use this as an axis to find the image P' of P. What does JP' equal? What does KP' equal? Using the results of Problem 7, what do you conclude about the relationship between P' and Q?

Fill in the following blanks to form a true statement:

If P and Q are two points which are _________ from a point _______ and which are also __________ from a point _________. then P and __________ are _____________ with respect to the axis through ____________ and _______________.

9. Given a point P, a line g, and a given number α. What is the set of all points H which are on the line g and which are such that PH= α ? Make a diagram to illustrate your answer. (Hint: There should be three cases.)

10. Suppose P is symmetric to P' with respect to an axis g. Let H be any point on g. What is the relationship between PH and P'H? Suppose you consider the line through P and P' as axis and let H' be the image of H is this axis. What do you know about PH' and P'H? Can you be sure that H' is on g? Can you give a good reason for your answer? If not, let's go further. Can you find a point G on f such that PG=PH? (Of course, we mean a point G distinct from H). How does P'G compare to PG. Why? How does P'G compare to P'H' and how does PG compare to PH'? Using the results of Problem 9, what do you conclude about G and H'?

Fill in the following blanks to form a true statement:

If P and P' are __________ with respect to an _______g, then g is _________ under reflection with respect to the __________ through _______ a d ______ as axis. The line g and the ______ through P and

Summary

A reflection is a distance-preserving mapping of the plane into itself which leaves fixed all points on one straight line. This line is called the axis.

All points on the axis are invariant under reflection. The distance between any two points is equal to the distance between their images. A point and its image are equidistant from any point on the axis. All points equidistant from two given points P and Q lie on the axis of symmetry of P and Q. The line joining a point and its image is invariant under reflection.

Two lines are perpendicular if each is invariant under reflection with respect to the other as axis. The line joining a point and its image is perpendicular to the axis. All lines perpendicular to the axis are invariant. If a line is the perpendicular bisector of the line segment joining P and Q, then the line is an axis of symmetry for the points P and Q.

Exercise III

In each of the following problems, show all construction lines and justify your construction by using the facts in the summary.

1. Construct the image of a point in a given axis.
2. Construct the axis of symmetry for two given points.
3. Construct the perpendicular bisector of the line segment joining two given points.

4. From a point outside a given line construct the perpendicular to the given line.

5. At a given point on a line construct a perpendicular to the line.

6. Given a point P and its image P' in a given axis, construct the image of a given point Q using only a straight edge.

7. Given a point P and an axis of reflection construct the image P' using only a band (the two parallel edges of a ruler).

When you look at an object, you see the object by the light rays which travel in a straight line from the object to your eyes. When you look in a mirror at the image of an object, you apparently see the image by the light rays which come from the image to your eyes. This is not really the case because the image does not exist in real space. You really see by the light rays going from the object to the mirror and then being reflected back to your eyes as in Figure 11.

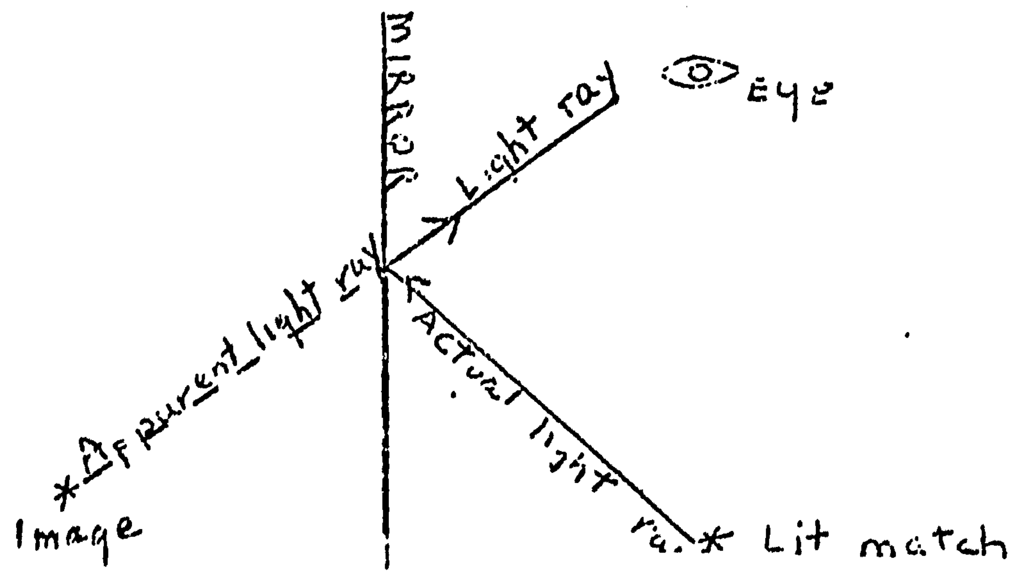
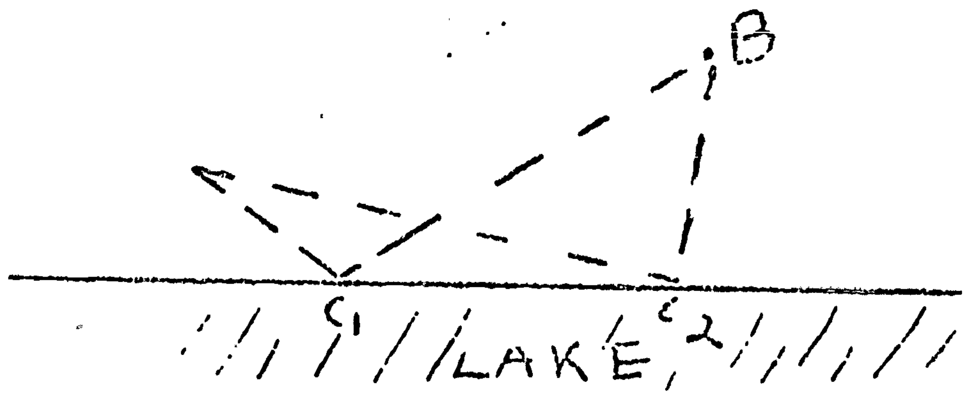


Fig. 11

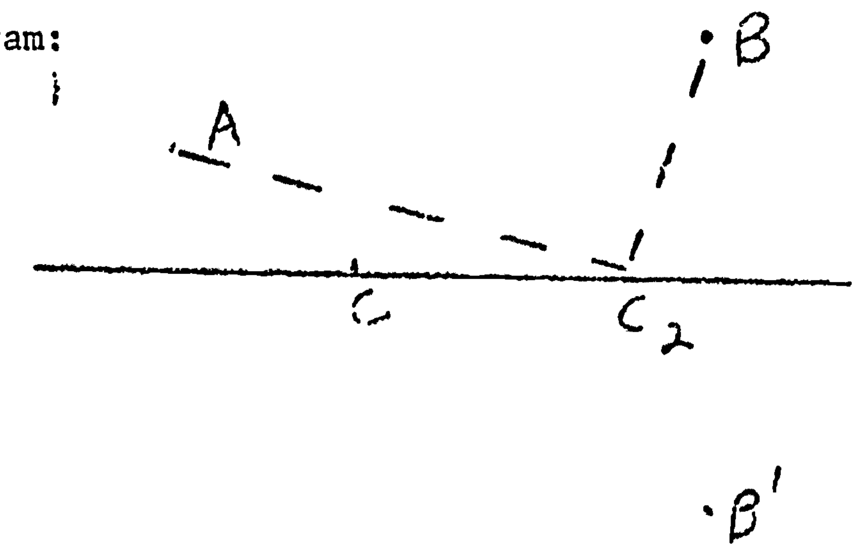
Exploratory Exercise

You are at the point marked A in the diagram below. You have an empty pop bottle which you want to fill in the lake and pour over your friend who is sleeping at the point marked B. Can you discover at what point in the lake you should fill your bottle so that the total distance from A to the lake to B is as short as possible?



Let's try to analyze this problem. Is it obvious that the distance AC_1+C_1B is shorter than the distance AC_2+C_2B? Is it obvious that neither C_1 nor C_2 is the best point?

Let's consider the image of B in the lake, thus getting the following diagram:



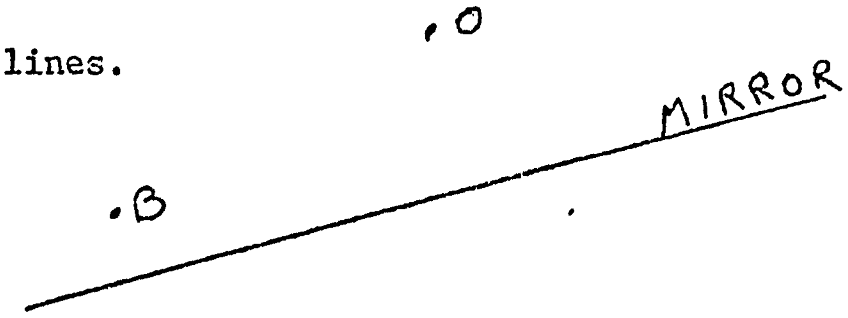
Draw the line AB' intersecting the edge of the lake in the point C. What is the relation between the distance CB and the distance CB'? What is the relation between C_2B and C_2B'? How does the sum of the distance AC_2 and C_2B compare to the sum of AC and CB'? Why? How does the sum of AC and CB compare to the sum AC_2 and C_2B? Why?

Describe how to find the path going from A to the lake to B which has the shotrest length. Give your reasons for the validity of the description.

Exercise IV

1. A French mathematician Pierre Fermat said, "Light travels along the path that takes the shortest time." Assuming that the speed of light is cc stant, explain this statement by reference to Fig. 11.

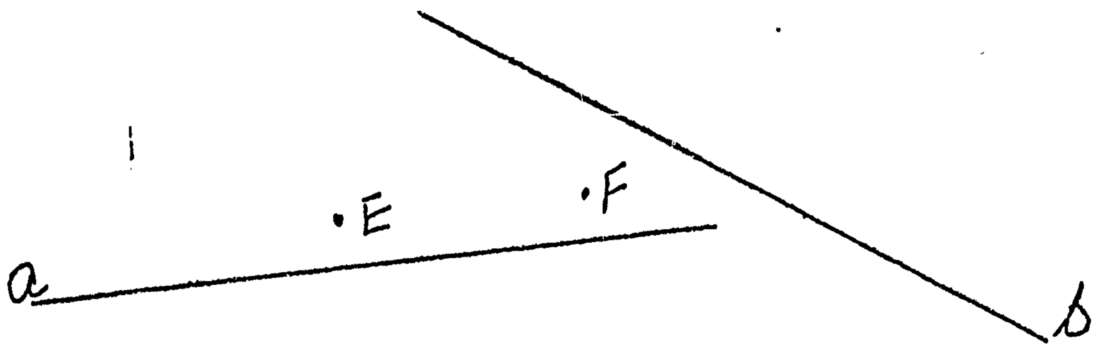
2. In the diagram below, construct the path of the light ray from the bulb at B to the mirror and then to the observer at O. Show all construction lines.



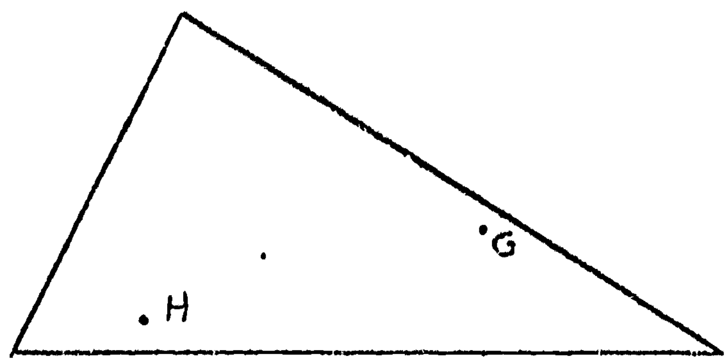
3. In problem 2, prove that the angle of incidence equals the angle of reflection.

4. In the exploratory exercise above, find the shortest path from B to the lake to A. How does this shortest path compare with the shortest path found in the exploratory exercise? Explain why this should be so.

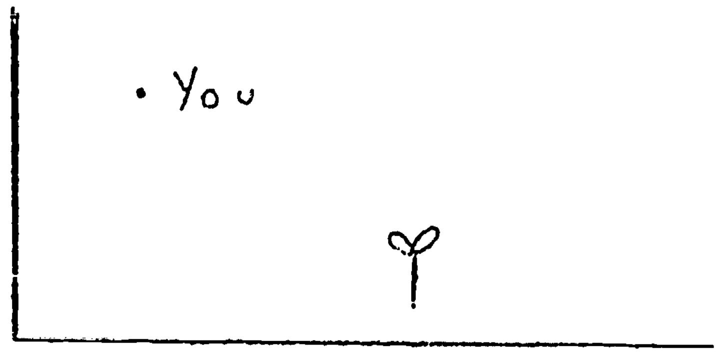
5. In the diagram below how could you find the shortest path that starts at E, goes to the line a, then to the line b, and ends at the point F? Justify your answer.



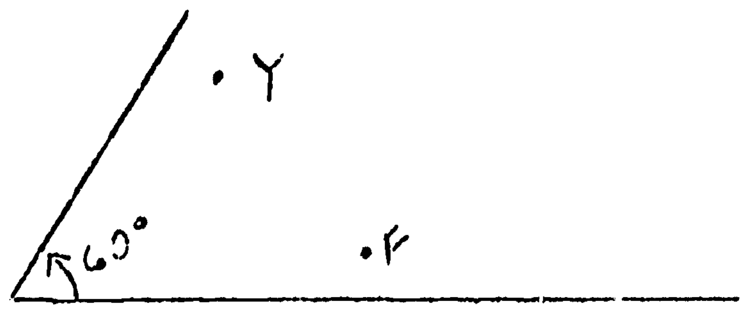
6. In the diagram below, how could you find the shortest path that starts at G, goes to each side of the triangle once and only once, and finishes at H?



7. Y u are standing near two right-angled mirrors. How many images of the flower can you see in the mirrors? Draw the path of the light ray for each image.



8. Do Problem 7 if the mirrors are 60^{o} apart.



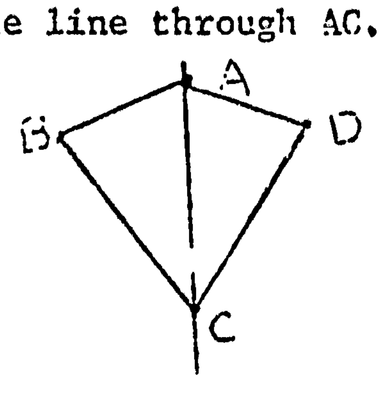
Make a conjecture about the number of images if the mirrors were 45^{o} apart, 30^{o} apart.

Exercise V

1. Find all sets of two points which are symmetrical with respect to an axis. Find all sets of 3 points, sets of 4 points, sets of 5 points which are symmetrical with respect to an axis.

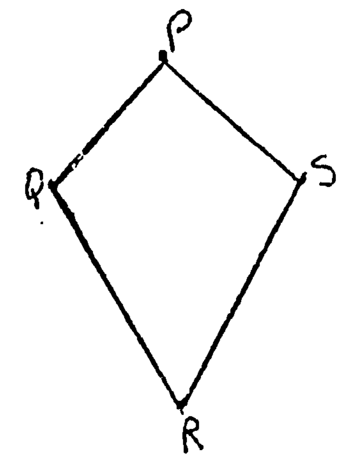
2. Suppose the set of three distinct points P, Q, and R is symmetric with respect to an axis. Show that the triangle whose vertices are P, Q, and R must be isosceles (have two equal sides) with one vertex on the axis. Suppose P is on the axis. How are the points Q and R related to the line which bisects the angle of the triangle at P?

3. Suppose the points A, B, C, D illustrated in the diagram are symmetric with respect to the line through AC. The figure ABCD is called a kite.

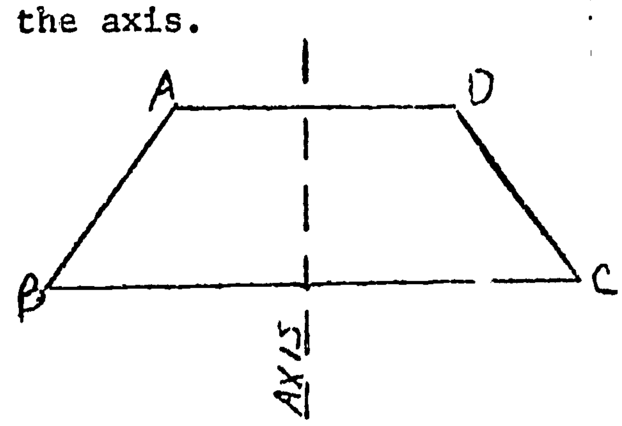


How does the length of AB compare with that of AD? Why? How does BC compare with CD? Why? What is the relation between the diagonals AC and BD? Why?

4. A kite is a four-sided, convex figure with two pairs of equal, adjacent sides. Thus, if PQ=PS and QR=QS, then PQRS is a kite. Show that the line through PR is an axis of symmetry for the kite. Show that the diagonals of the kite meet at right angles. Show that PR is the perpendicular bisector of QS. Show that PR bisects the angles at P and R.



5. Suppose the points A, B, C, D illustrated in the diagram are symmetric with respect to the axis.



Which two sides of the figure ABCD are equal? Why? Which two sides of the figure ABCD are parallel? Why? The figure ABCD is called an <u>isosceles trapezoid</u> -- isosceles because two sides are equal, trapezoid because two sides are parallel.

Where is the midpoint of the side AD located? The midpoint of the side BC? Where do the diagonals AC and BD intersect? Why? If the sides AB and DC are extended until they meet in a point E, what can you say about the location of E? What can you say about the triangle EAD? The triangle EBC? Why? What can you say about the line connecting the midpoints of sides AB and DC?

A <u>reflection</u> of a <u>sphere</u> with respect to a great circle is a distance-preserving mapping of the sphere into itself which leaves the points of the great circle fixed. We call this great circle the axis of the reflection.

Exercise VI

By a line on a sphere we shall mean a great circle. The length of a line segment PG is the distance from P to G measured along the great circle joining P and G.

1. Do Problems 3 and 4 in Exercise II if the diagram refers to points on a sphere.

2. Do Problem 5 in Exercise II if the points mentioned are on a sphere, not on a plane.

3. Do Problems 6 and 7 in Exercise II if all the points mentioned are on a sphere and not on a plane.

4. Find all configurations of 3 points on a sphere which are symmetric with respect to a line.

5. Find all configurations of 4 points on a sphere which are symmetric with respect to a line.

6. Refer to Problem 5 in Exercise V. Suppose the four points A, B, C, D are on a sphere and have a common axis of symmetry. Are the lines AD and BC parallel? Do there exist on the sphere parallel lines (lines, that is, which do not meet no matter how far extended)? What is the relationship between the lines AD and BC?

Two Perpendicular Axes of Symmetry

Exercise VII

1. Suppose a kite ABCD has two perpendicular axes of symmetry. Show that all sides of the kite must have equal length.

2. A four-sided figure with all sides of the same length is called a <u>rhombus</u>. Show that a rhombus must have two perpendicular axes of symmetry. Show that the diagonals of a rhombus are perpendicular to each other. Show that the diagonals of a rhombus bisect each other. If AB and DC are opposite sides of a rhombus, draw the line connecting the midpoints of AB and DC. Draw the line connecting the midpoints of AD and BC. Show that these lines intersect at the intersection of the diagonals.

3. Suppose a four-sided figure has perpendicular diagonals. Is it necessarily a rhombus or a kite? Suppose a four-sided figure has diagonals that bisect each other. Is it necessarily a rhombus or a kite? What must you know about the diagonals of a four-sided figure to be sure it is a kite, to be sure it is a rhombus?

4. Suppose an isosceles trapezoid ABCD has two perpendicular axes of symmetry. Show that all angles of the figure must be right angles.

5. A four-sided figure with all angles right angles is called a <u>rectangle</u>. Show that a rectangle must have two perpendicular axes of symmetry. Must the diagonals of a rectangle be perpendicular to each other? Why? Must the diagonals of a rectangle bisect the angles at the vertices?

6. Given two straight lines a and b intersecting at a point O. Show that this figure has two axes of symmetry. Show that the axes of symmetry are perpendicular to each other. Show that these axes bisect the angles between the straight lines.

Central Symmetry

Suppose lines a and b in Figure 12 are perpendicular. Consider the image P' of a point P after successive reflections in lines a and b. Does it make any difference if we reflect first in a and then in b, or if we reflect first in b and then in a?

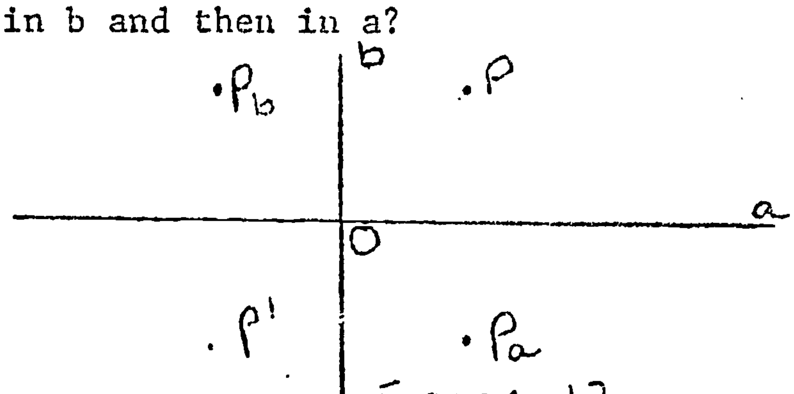


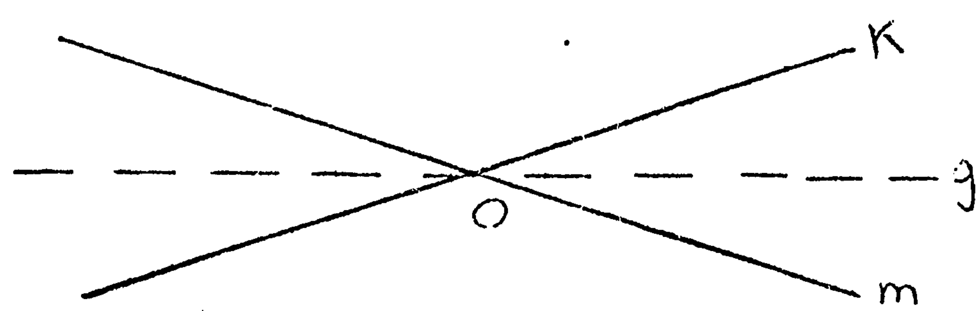
Figure 12

Give reasons for your answer. We say that P' is the image of P under a central symmetry with respect to the point O, the intersection of lines a and b. We also say that P' is obtained by reflecting the point P with respect to the point O.

Exercise VIII

1. How does the length OP above compare to the length OP'? Why?

2. In the following diagram suppose g is the axis of symmetry for the lines k and m meeting at O.



Show that k and m have another axis of symmetry going through O. Show that this axis is perpendicular to g.

3. In Fig. 12 show that a and b are axes of symmetry for the lines through OP and through OP_a. Show that P' is on the line through OP and that OP=OP'.

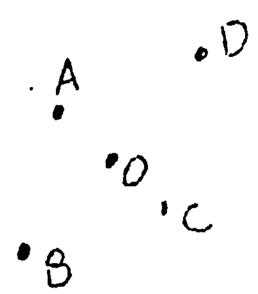
4. Given a point P, construct the image of P in a point O. Given two points A and B construct a center of symmetry O for the two points.

5. Suppose a figure is symmetric with respect to a point O and with respect to a line g passing through O. Show that the figure must also be symmetric with respect to a line h perpendicular to g at O.

6. What are all the points invariant with respect to a reflection in the point O? What are all lines invariant with respect to this reflection?

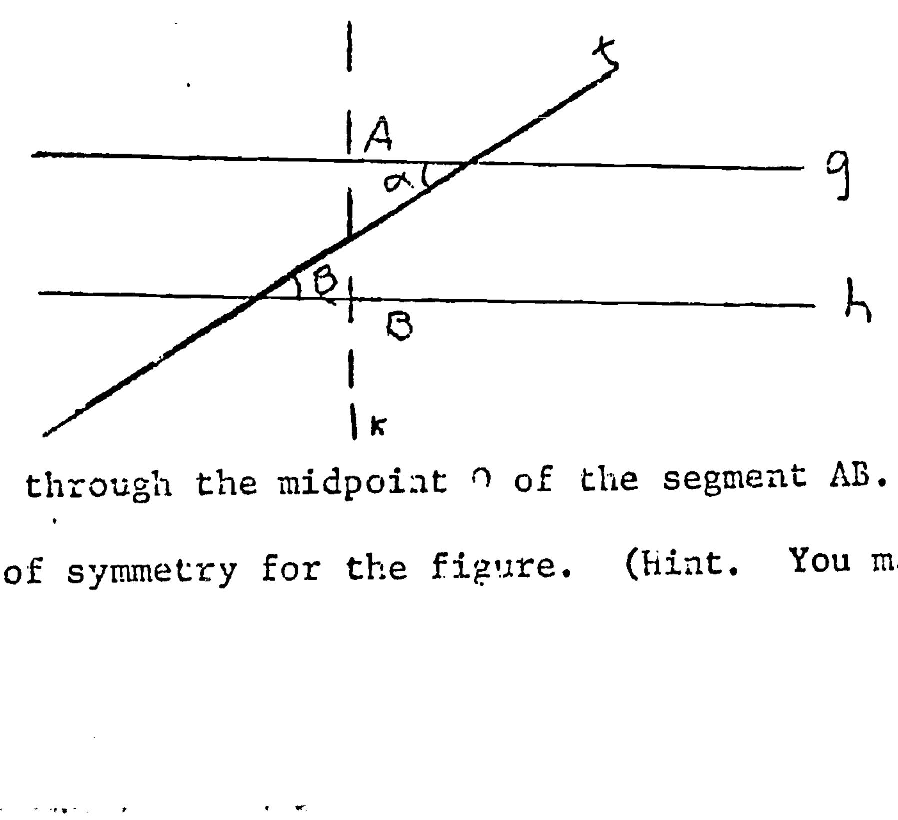
7. Find all configurations of 3 points which are symmetric with respect to a point. Can a triangle have a center of symmetry? Why?

8. Suppose the points A, B, C, D in the following diagram have the point O as the center of symmetry:



What is the image of A in O? What is the image of B in O? How does the length AD compare with BC? How does the length AB compare with CD? Draw a line through O perpendicular to the segment AD. What is the angle between this line and the line segment BC? What is the relation between the line through AD and the line through BC? What is the relation between the line through AB and the line through CD?

9. In the diagram below lines g and h have a common perpendicular k cutting g and h in A and B, respectively.



Suppose t is a line through the midpoint O of the segment AB. Show that O is a center of symmetry for the figure. (Hint. You may want to draw the line m through O perpendicular to AB). What can you say about the relation between the angles α and β ?

10. Given two lines on a plane, how many common perpendiculars may they have? Given two lines on a sphere, how many perpendiculars may they have?

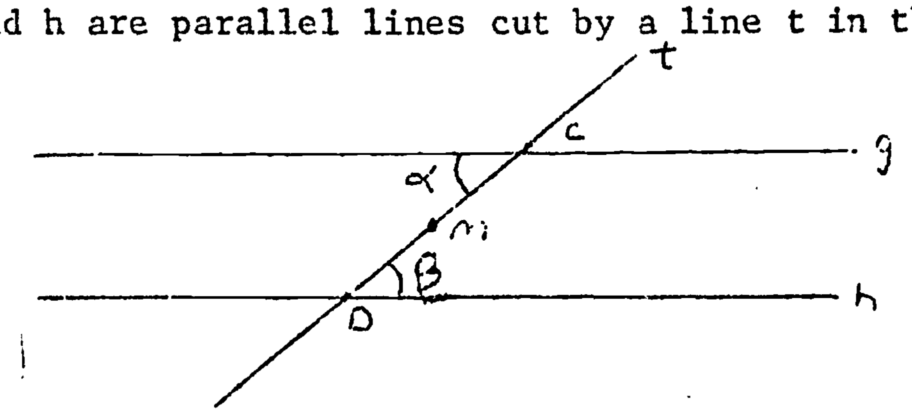
Parallel Lines and Parallelograms

Two lines in the plane are said to be parallel if they never meet. We shall use instead the following definition: Two lines are parallel if they have a common perpendicular. In the plane it follows that through every point in the plane there is a line perpendicular to both lines.

A four-sided figure whose opposite sides are parallel is called a parallelogram.

Exercise IX

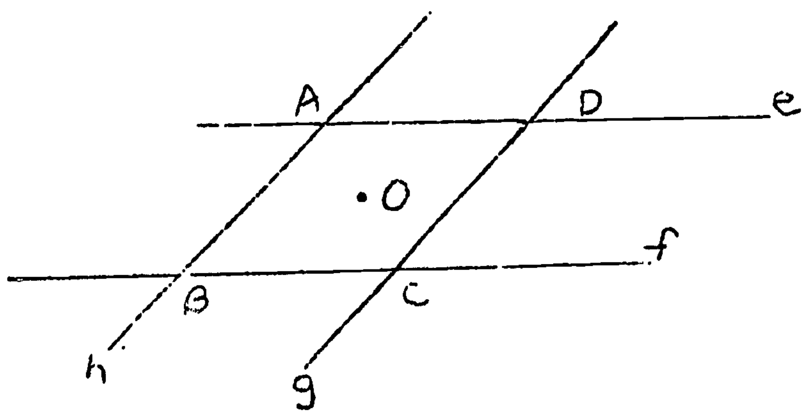
1. Suppose g and h are parallel lines cut by a line t in the points C and D.



Show that this figure is symmetric with respect to the midpoint M of the segment CD. (Hint. See Problem 9 in Exercise VIII).

2. The line t in the diagram for Problem 1 is called a transversal for the parallel lines and the angles α and β are called alternate interior angles. Show that $\alpha = \beta$.

3. Suppose ABCD is a parallelogram and suppose O is the midpoint of the diagonal AC.



Is AC transversal for the lines e and f? Then O is a _______ of _______ for lines e and f. Is AC a transversal for the lines g and h? Then O is a _________ of _________ for the lines ________ and ___________. What is the image of A in O? What is the image of B in O? Why? Where do the diagonals AC and BD intersect? Why? How does OB compare to OD? Why?

Fill in the blanks to make true statements.

The intersection of the __________ is a __________ of symmetry for a parallelogram.

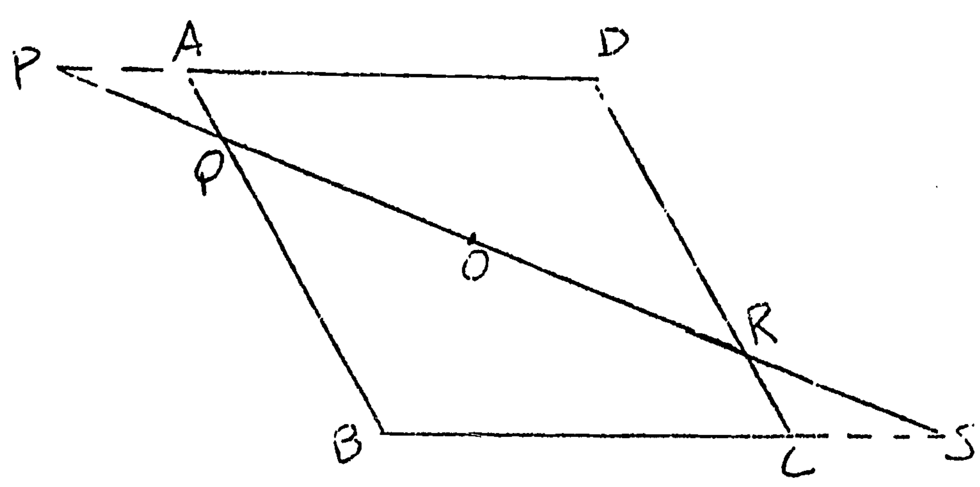
4. Show that the diagonals of a parallelogram bisect each other. Show that opposite sides of a parallelogram are equal.

5. Show that a four-sided figure which has a center of symmetry must be a parallelogram.

6. Show that if AD=BC and AD is parallel to BC then ABCD is a parallelogram. (Hint. Let O be the midpoint of AC; show that O is a center of symmetry.)

7. Show that if AD=BC and AB=DC, then ABCD is a parallelogram. (Hint. Same as in Problem 6).

8. In the diagram below, ABCD is a parallelogram and O is the intersection of the diagonals. Show that PQ=RS.



Summary

If Q is the image of P under successive reflections in two perpendicular axes intersecting in O, then Q and P are symmetric with respect to the <u>center of symmetry</u> O. We say that Q is the image of P with respect to the point O.

A central symmetry is a distance-preserving mapping of the plane into itself which leaves invariant every line through the center.

If P and Q are symmetric with respect to the point O, then O is on the line through P and Q and OP=OQ.

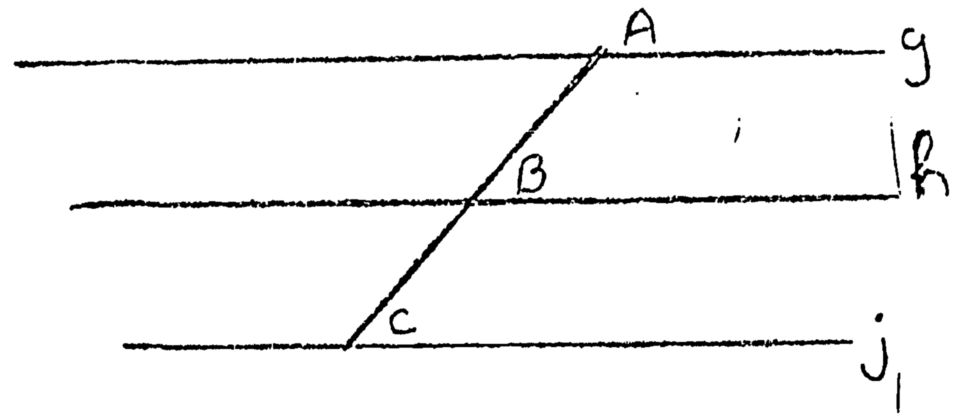
The image of a line g, not going through O, with respect to O, is a line h parallel to g and the distance from O to g is equal to the distance from O to h.

The intersection of the diagonals of a parallelogram is a center of symmetry for the parallelogram.

If a figure has a center of symmetry and an axis of symmetry passing through this center, then the line perpendicular to the original axis at the center of symmetry is also an axis of symmetry.

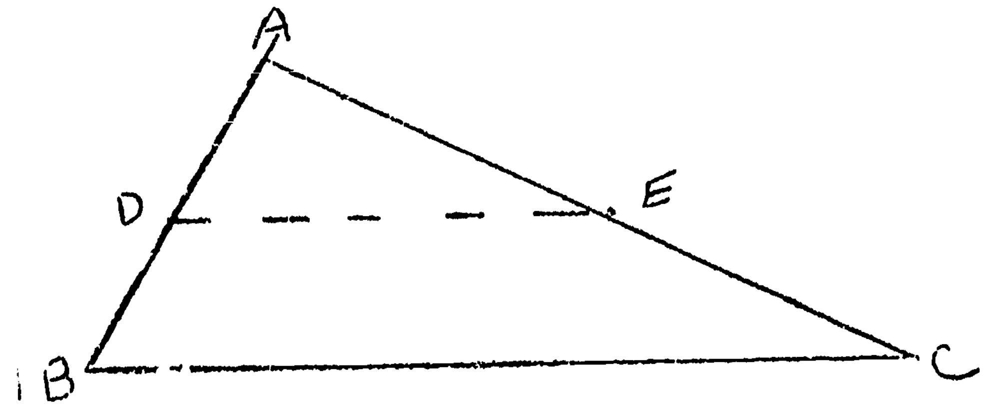
Exercise X

1. Suppose g, h, and j are parallel lines and h is half way between g and j (see diagram).



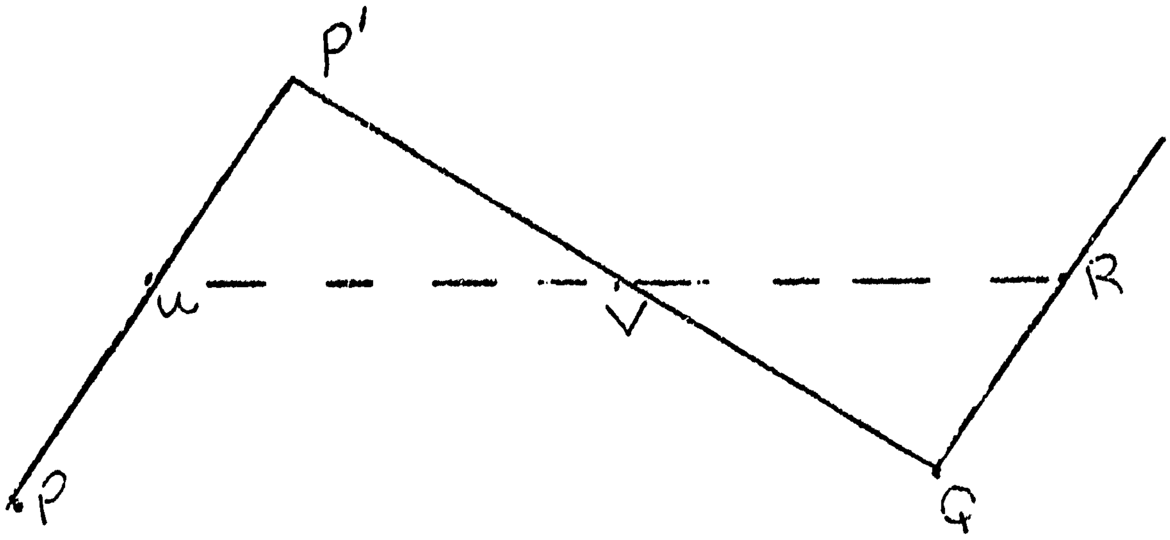
How are g and j related to B? Why? How does AB compare to AC? Why?

2. In triangle ABC below, AD=DB and DE is parallel to BC.



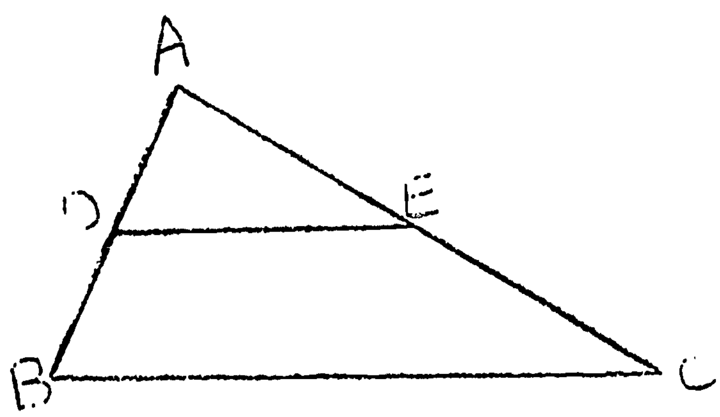
Show that AE=EC. (Hint. Draw at A a parallel to BC and use Problem 1).

3. Suppose P is reflected through the point U into the point P' and then through V as center into the point Q (see diagram).



3. (cont.) What is the relation between PU and P'U, between P'V and VQ? Why? What is the image of the line through PP' in the center U? Show that the image of this image is the line through Q parallel to PP'. Suppose the line through UV intersects this parallel in R. Why does the image of U in V have to be on QR? Why does the image of U in V have to be on the line through UV? Why is R the image of U in V? Why is UV=VR? What is the image of U in U as center and then in V as center? Why is PU=QR? Show that PURQ is a parallelogram. (Hint. Exercise IX, Problem 6). Show that UV=½PQ.

4. In triangle ABC suppose D is the midpoint of side AB and E is the midpoint of AC.

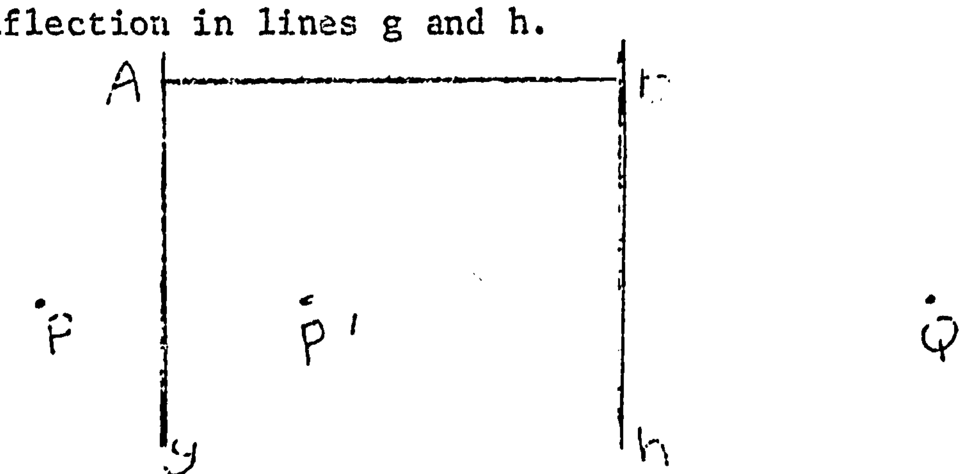


Show that DE is parallel to BC and that DE=½BC. (Hint. Use Problem 3).

Two Parallel Axes of Symmetry

Exercise XI

1. Suppose lines g and h are parallel. Suppose Q is the image of P after successive reflection in lines g and h.



How is the line PQ related to the lines g and h? Why? Suppose AB is the length of the common perpendicular to g and h. How is PQ related to AB? Why? Draw a diagram to illustrate the case when P is between the lines g and h. Are your answers to the preceding questions still correct? Suppose P is very far to the right of Q. Would your answers still be correct?

2. Which points P are invariant under the mappings described in Problem 1? Which lines are invariant?

3. Suppose P is reflected first in g and then in h, giving an image Q. Would you get the same image if you reflected P first in h and then in g? Is there any point P for which the two images are the same? How are these two images related to the original P?

4. Suppose you replace the lines g and h by two other parallel lines g' and h' which have the same common perpendicular as g and h and which are the same distance apart as g and h are. Compare the image of P in g and h with the image of P in g' and h'. Does the mapping depend on the distance between the parallel lines? Does the mapping depend on the common perpendicular of the two lines? Does the mapping depend on the exact location of the two lines?

5. Fill in the blanks to make a true statement.

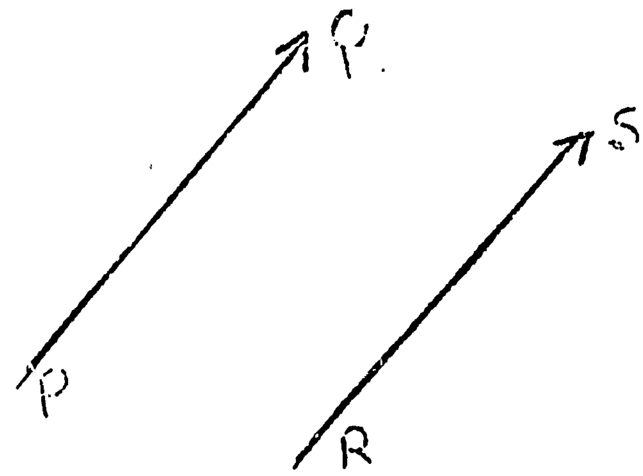
When a point is successively reflected in two __________ lines g and h, the image Q is a point such that the __________ through P and Q is __________ to __________ and __________. The length __________ is __________ the distance between __________ and __________.

6. In the diagram for Problem 1, suppose C is the image of A by successive reflections in g and h. Show that ACQP is a parallelogram. Would this statement be correct if you started with P to the right of C?

Summary

Suppose a plane is mapped into itself by successive reflections in two parallel axes which are a distance d apart. If P goes into Q under this mapping, the length PQ=2d and the line PQ is a common perpendicular to the two axes. We call such a mapping a translation of the plane.

If P maps into Q and R into S under a translation, then PQ=RS and the line through PQ is parallel to the line through RS. Since the

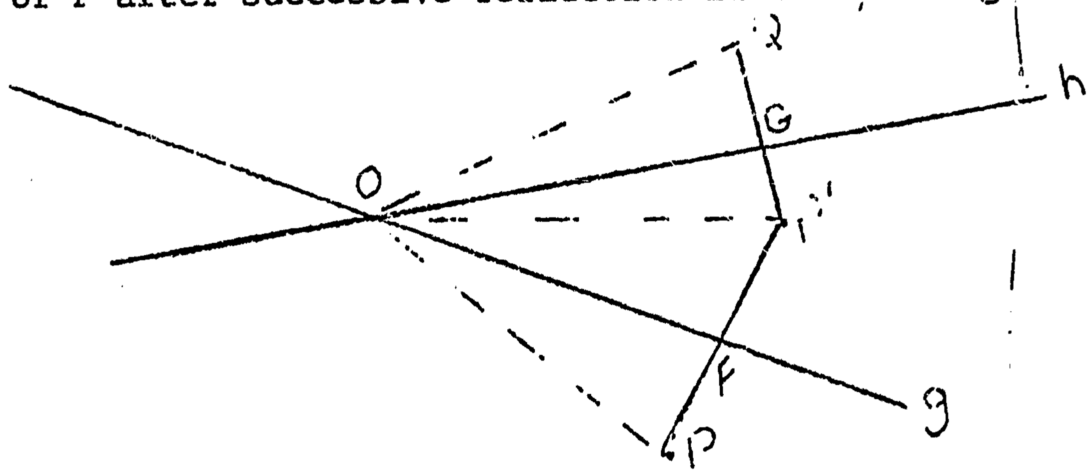


translatinn is completely specified as soon as the image of one point is known, we may represent the translation by the vector PQ. Note that PQ=RS because each represents the same translation.

Two Intersecting Axes of Symmetry

Exercise XVI

1. Suppose lines g and h intersect in a point O. Suppose Q is the image of P after successive reflection in the lines g and h.



a. What is the image of O after successive reflections in g and h? Why?

b. What is the image of the line OP after successive reflections in g and h? Why?

c. Suppose F is the intersection of PP' with g and G is the intersection of P'Q with h. How does the angle POF compare with the angle FOP'? Why? How does the angle P'OQ compare with the angle GOQ? Why? How does the angle POQ compare with the angle FOG=angle between g and h? Why?

2. Suppose that P in the above diagram is located in the region where P' is marked.

a. Can the argument of Problem 1 still be carried out?

b. Is the result that P is rotated through an angle twice the angle of FOG still correct?

3. Fill in the blanks to make the following a true statement:

The image of a __________ after two successive __________ in axes intersecting at O is obtained by __________ the point around __________ as center through an angle which is __________ the angle between the __________.

4. Suppose that P is first reflected in h and then in g. What rotation is that equal to?

b. Suppose h and g are perpendicular. What is the rotation obtained by first reflecting in g and then in h?

c. A central symmetry can be obtained by a rotation through what angle?

5.a. What points are fixed under a rotation? What lines are fixed under a rotation?

b. What is the image of a line through O under rotation around O? What is the image of a line not through O?

6. Suppose a figure is invariant under rotation about O through 120°. Prove that it must be invariant under rotation through 240°. (Hint. Suppose the image of P under the 120° rotation is Q and suppose the image of Q under that rotation is R. What is the mapping that will map P into R?)

Summary

Successive reflections in two axes intersecting at 0 produces a rotation around 0 through an angle equal to twice the angle between the axes.

The following are distance-preserving mappings of the plane into itself: 1) reflections about a line, 2) reflections through a point, 3) translations, and 4) rotations.

Notation

To collect and organize our results, it is useful to have a systematic notation. Let us use capital letters such as A, B, P, etc. for points and lower case letters such as g, h, k, etc. for lines. We shall use P^g (read "P to the g") for the image of P reflected in g.

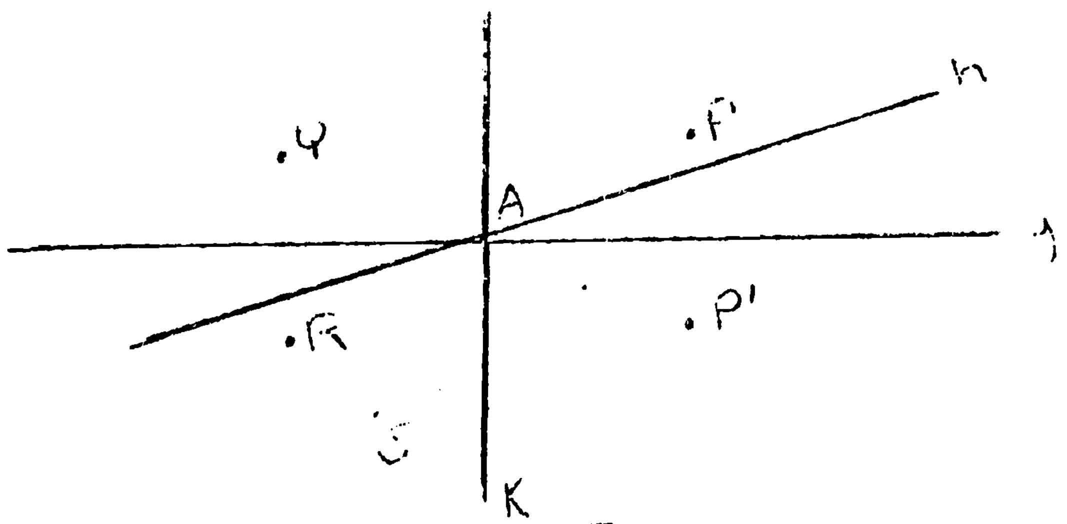


Fig 13

Thus in Figure 13 $P^g=P'$. Similarly, $Q^h=S$. We shall use P^A for the image of P reflected in the point A. Thus in Figure 13,

$$P^A=R, \quad Q^A=P', \quad (P')^k=R, \quad (P')^A=Q.$$

We may indicate successive reflections as follows:

$$(P^g)^k = (P')^k = R$$

$$(P^k)^h = Q^h = S.$$

We shall also use the following notation:

$$P \xrightarrow{g} P' \xrightarrow{k} R$$

Read "P is mapped by g into P'; P'is mapped by k into R."

Exercise XIII

1.a. Does $(P^g)^k = (P^k)^g$ for every point?

b. Does $(P^g)^h = (P^h)^g$ for every point?

c. Does $(P^g)^g = P$ for every point?

Whenever your answer is no, give an example. Whenever your answer is yes, give a proof.

2. If an and b are any two lines, let us write $(P^a)^b = P^{ab}$.

a. What does P^{aa} equal?

b. Does $P^{ab} = P^{ba}$ for every pair of lines a and b? Why?

c. Suppose $P^{ac} = P$ for every point P. What can you say about line a and line c? Why? Does $(P^{ab})^c = (P^a)^{bc}$ for every point P? Why?

3. Suppose $P^{ab} = P^{ba}$ for every point P. What can you say about the lines a and b? What does P^{AA} equal? Suppose $P^{AB} = P$ for every point P. What can you say about A and B? Suppose $P^{ab} = P^{A}$ for every point P. What can you say about lines a and b? Fill in the blanks to make a correct statement:

$$P^{aA}=P^{\_\_\_}, \quad P^{Ab}=P^{\_\_\_}.$$

4. Suppose e and f are perpendicular lines intersecting in D. For each of the following, state whether it holds for every point P in a plane and give a reason for your answer:

a. $P^{ef}=P^{fe}$

b. $P^{ef}=P^{f}$

c. $P^{fD}=P^{e}$

d. $(P^{fD})=P^{e}$

e. $(P^{ef})^{e}=P^{f}$

f. $P^{ee}=P$

g. $((P^{ef})^{e})^{f}=P$

Mappings

If we apply a distance-preserving mapping of the plane into itself and then apply another distance-preserving mapping, we end up with a distance-preserving mapping from the original state to the final state. We call this successive application of mappings a composition (Latin for "putting together") of mappings. For example, the composition of reflections about perpendicular axes is a reflection through the origin; the composition of two rotations about the same point is also a rotation about that point.

Exercise XIV

Fill in the blanks to make correct statements.

1. The composition of two reflections about parallel axes is a __________ whose magnitude is __________ the distance __________ the __________ and whose direction is the direction of the __________ __________ of the axes.

2. The composition of two reflections about perpendicular axes is a __________ symmetry about the point of __________ of the __________.

3. The composition of two reflections about intersecting axes is a __________ about the __________ of __________ of the __________ through an angle which is __________ the angle __________ the axes.

4. A __________ symmetry is a __________ through an angle of __________ degrees.

5. The composition of central symmetries about two distinct points is a __________ whose magnitude is __________ the distance __________ the __________ and whose direction is the direction of the __________ joining the __________. (Hint. Replace each central symmetry by the composition of reflections through perpendicular axes. Use the line joining the points as an axis.)

Instead of studying the image of a point under a mapping, we want to study the mapping as an entity by itself and also we want to study the composition of mappings. Before doing so, we introduce some simplifications in our notation. We have found that, if a and b are perpendicular lines,

$$P^{ab}=P^{ba}$$

for every point P in the plane. We shall write this as follows:

$$ab=ba$$

but the meaning is still that

$$P^{ab}=P^{ba}$$

Similarly, we found that

$$P^{aba}=P^{b}.$$

For every point P in the plane, we may write this as

$$aba=b.$$

We shall call this the mapping notation and the original notation we shall call the point notation. Similarly, if a and b are perpendicular lines intersecting in a point C, then we know that

$$P^{ab}=P^{C}$$

for all points P. In the mapping notation we write this as

$$ab=C.$$

We need one further notation. We know that

$$P^{aa}=P$$

for every point P in the plane. How can we write this in the mapping notation? We would like to write aa=something, but what is "something"? The "something" is a mapping which maps P onto P for every point P of the plane. Since this mapping leaves every point P in the identical position it was originally, we call this mapping the <u>identity</u> mapping and we symbolize it be the letter i. Thus,

$$P^{i}=P$$

for all points P and we then have

$$aa=i.$$

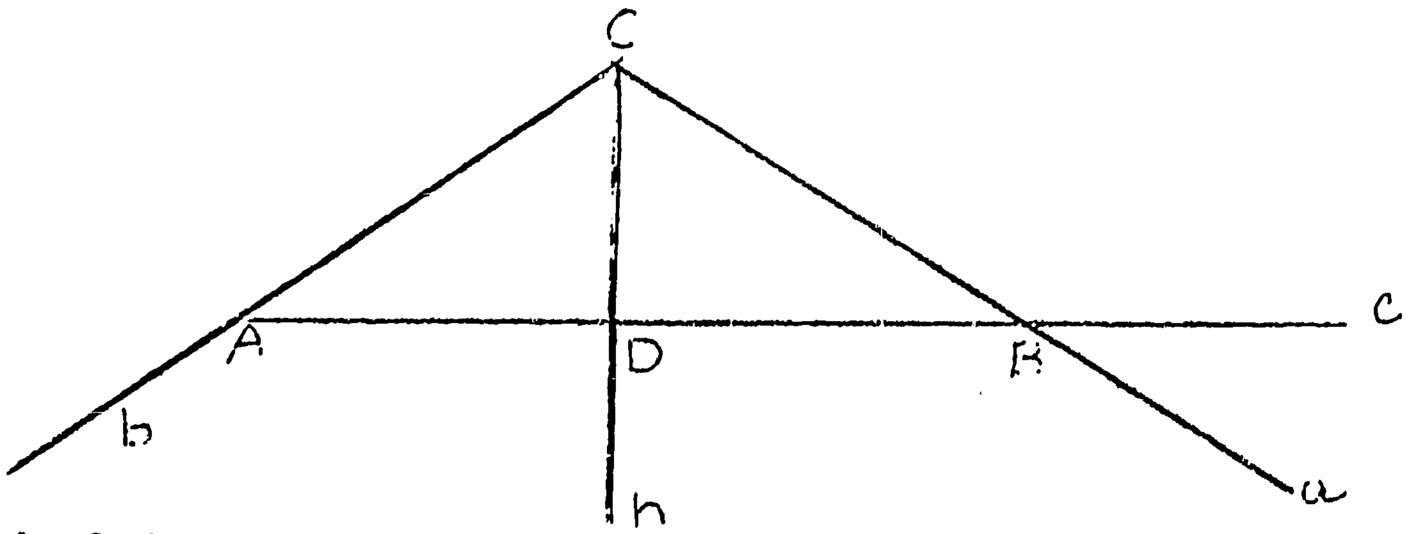
Exercise XV

1. Write the following relations in point notation:

a. bab=b b. bA=a c. ab=A d. ab=ba

2. Write each of the relations stated in Exercise XIII, Problem 4 in the mapping notation.

3. In the following diagram, triangle ABC is isosceles with AC=BC and D the midpoint of AB.



For each of the following relations involving mappings, state whether it is correct or incorrect and give reasons for your answers:

a. hh=i b. BB=i c. ch=D d. bc=A e. ab=ba

f. ch=hc g. ah=ha h. ahbh=i i. cb=ac j. ADBD=i

5. Prove that the composition of a and b is commutative, that is, ab=ba if, and only if, a and b are reflections in perpendicular axes.

Groups

Exploratory Exercise

Suppose a figure is symmetric with respect to line a and with respect to another line b. Let us find out whether the figure must have other symmetries. If Z is a point of the figure, then why are Z^a and Z^b points of the figure? Why are $(Z^a)^a$ and $(Z^a)^b$ points of the figure? What is $(Z^a)^a$ equal to, for all points Z? For which points Z does $Z^{ab}=Z$? For which points Z does $Z^{ab}=Z^a$, does $Z^{ab}=Z^b$? We see that if Z is not on the line a and not on the line b, then the points Z, Z^a, Z^b, and Z^{ab} are all different. Do you now know another symmetry of the figure?

We have reached the following conclusion:

If a figure is symmetric under the mapping a and symmetric under the mapping b, then it is also symmetric under the mappings ab and ba.

Write a similar conclusion if a figure is invariant under a mapping A and a mapping c.

Let us continue investigating the set of symmetries of a figure. Suppose a figure is invariant under a rotation of 90° around a point. Must it be invariant under a rotation of 120°, 135°, 180°? What other symmetries must it have?

Suppose a figure is invariant under a translation of 2 units upward? Must it be invariant under a translation 3, 4, 5, 6 units upward? Must it be invariant under any downward translations? What are all symmetries of this figure?

From this exercise we draw the following conclusions:

1. If a figure is invariant under two mappings, it is also invariant under the composition of the mappings.

2. If a figure is invariant under any mapping, it is also invariant under any repeated composition of the mapping with itself.

3. If a figure is invariant under any mapping, it is also invariant under the inverse mapping, that is, the mapping which interchanges the original and the image point. For example, the inverse mapping to a rotation of 30° in the clockwise direction is a rotation of 30° in the counter-clockwise direction. The inverse mapping to a translation of 5 units in a certain direction is a translation of 5 units in the opposite direction.

What is the inverse mapping to a reflection, to a central symmetry?

The set of all symmetries of a given figure is called the symmetry group of that figure. Note the identity mapping belongs to every symmetry group.

Exercise XVI

1. Give reasons why each of the following sets is or is not a symmetry group:

a. i

b. i, a, b, C, where a and b are perpendicular lines intersecting in C

c. i, a, b, ab, ba, where a and b are parallel lines

cont. next page

d. i, a

e. a, b, C, where a and b are perpendicular lines intersecting in C

f. i, a, b, ab, if a and b are perpendicular lines

g. i, a, b, ab, aba, abab, if a and b are lines intersecting at sixty degrees.

2. What is the symmetry group of an equilateral triangle?

3. What is the symmetry group of a rectangle, of a square?

4. What is the symmetry group of a regular hexagon, a regular octagon?

5. Do the set of reflections about any line in the plane form a symmetry group?

6. Do the set of all possible rotations around a fixed point form a symmetry group?

7. Do the set of all translations in the plane form a symmetry group?

A symmetry group is a set of mappings which

1. con ains the identity mapping
2. contains the composition of any two mappings of the set
3. contains all compositions of a mapping of the set with itself.

Notice that the composition of mappings is always associative, that is, if a, b, and c are three mappings, then

$$(ab)c=a(bc).$$

Geometry through Symmetry for Liberal Arts students at Berkeley

The following notes were written in an attempt to provide a rigorous foundation for the material presented to the eighth grade students at Stanford. The material was tried out in a class at the University of California at Berkeley. The class consisted of Liberal Arts students who had a minimum of mathematical preparation and were not planning to continue their study of mathematics.

The concept "point" is undefined.

Def. A space is a collection of points.

Axiom 1. To every pair of points in the space is assigned a positive number which has the properties a), b), and c) specified below.

The number assigned to the pair of points A and B is called the distance from A to B and is denoted by $d(A,B)$.

Properties of distance:

a) $d(A,B) = d(B,A)$

b) If A and B are distinct points, then $d(A,B)$ is greater than ($>$) zero. The number $d(A,B) = 0$ if, and only if, A is identical with B.

c) For any points A, B, and C the triangle inequality

$$d(A,C) + d(C,B) \geqq d(A,B)$$

holds.

Def. For any given points A and B the set of points X such that

$$d(A,X) + d(X,B) = d(A,B)$$

is called the line segment joining A and B. This line segment is denoted by $\overline{AB}$.

Problems

1. If A is the same as B, what is $\overline{AB}$?

2. A space has three distinct points labelled X, Y, and Z. If $d(X,Y) = 2$ and $d(X,Z) = 4$, how small and how large can $d(Y,Z)$ be?

3. A space has four distinct points labelled E, F, G, and H. Suppose that $d(E,H) = 1$, $d(F,H) = 3$, and $d(G,H) = 6$. Find the largest and smallest possible values for each of the distances $d(E,F)$, $d(F,G)$, and $d(G,E)$. Show that if these last three distances have each their smallest possible values, then F is on the segment joining E and G. Try other values for the first three distances, but always keeping them in the same relative magnitude, and see if F must be on $\overline{EG}$.

4. A space has four points labelled J, K, L, and M. Let us write JKL to mean that K is on $\overline{JL}$.

   Consider the following statements:

   a) If JKL and JKM, then JLM.

   b) If JKL and JLM, then JKM.

   c) If JKM and JLM, then either JKL or JLK.

   Test each statement for its truth or falsity by picking values for the distances between the points in the space. Make a conjecture as to the truth or falsity of each statement.

To be sure that a line segment has enough points, we use the following:

Axiom 2. Given any real number k between zero and one inclusive, and given points A and B, there exists a single point C in $\overline{AB}$ such that $d(A,C) = k\, d(A,B)$.

To be sure that every line segment can be extended into a larger segment, we use the following:

Axiom 3. For any points A and B, there exists a point C such that B is in $\overline{AC}$ and a point D such that A is in $\overline{DB}$.

Def. The segment $\overline{AB}$ <u>is contained in</u> the segment $\overline{XY}$, or, the segment $\overline{XY}$ <u>contains</u> the segment $\overline{AB}$ if every point in $\overline{AB}$ is also a point in $\overline{XY}$.

Lemma 1. If C and D are the points mentioned in Axiom 3, then $\overline{AB}$ is contained in $\overline{AC}$ and also in $\overline{DC}$.

Problems

5. Prove Lemma 1.
6. Suppose the space is the set of points on the surface of a sphere. Are the axioms satisfied by this space? What is a line segment in this case?

From Axiom 1 we can prove

Lemma 2. If X is in $\overline{AB}$, then $\overline{AX}$ is contained in $\overline{AB}$.

Lemma 3. If X is in $\overline{AB}$ and X is in $\overline{AX}$, then $\overline{YX}$ is contained in $\overline{AB}$.

From Axiom 2 we can prove

Lemma 4. If X and Y are in $\overline{AB}$, then either X is in $\overline{AY}$ or Y is in $\overline{AX}$.

Theorem 1. If X and Y are in $\overline{AB}$, then $\overline{XY}$ is in $\overline{AB}$.

Def. Two line segments <u>intersect</u> or <u>cut</u> in a point P if P is in both line segments.

Def. The <u>intersection</u> of two line segments is the set of all points in which the segments intersect.

The intersection of two line segments may be empty, that is, there is no point in which the segments intersect, or the intersection may be one point, or the intersection may contain an infinite number of points.

Def. The <u>union</u> of two segments is the set of points which are either in one segment or the other, or in both segments.

Problems

7. Draw illustrations on the plane of two segments which intersect in no point, in one point, in an infinite number of points. Can there exist on the plane two segments which intersect in exactly two points? Answer the same question for the surface of a sphere.

8. Draw illustrations on the plane of two segments whose union is not a segment and of two segments whose union is a segment. What is the intersection of these two segments in each case? Make a conjecture about how the intersection of two segments relate to the question of whether the union of two segments is a segment.

9. On the surface of the sphere does there exist two segments whose intersection has an infinite number of points and yet the union of these segments is not a segment? Before giving your answer, consider, on the surface of the earth, the segment connecting the South Pole with Seattle which is located at 53° N and 122° W, and consider the segment connecting San Francisco located at 37°N, 122°W with Magnitogorsk located at 53°N, 58°E.

Def. The <u>line</u> through the points A and B is the union of all line segments that contain the segment $\overline{AB}$.

Axiom 4. Two distinct lines intersect in at most one point.

Theorem 2. Through two distinct points there exists one and only one line.

Problems

10. Is the equator a line according to the above definition? If yes, give two points which determine the line.

11. Describe the line through Seattle and San Francisco on the earth's surface. Does this line contain the entire 122°W meridian and the entire 58°E meridian?

12. Consider a cylinder with its axis vertical. Describe the line on the cylinder through two points on the cylinder which are located on the same vertical generator. What is the line through two points which are on the same horizontal plane? Pick two points at random on the cylinder and describe the line through them.

13. Is a great circle on a sphere always a line through any two of its points? Is Axiom 4 satisfied by the lines on the surface of a sphere?

Def. If the three points A, B, and C are not all on one line, the union of the segments $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ is called the triangle ABC.

Def. The points A, B, and C are called the vertices of triangle ABC and the segments $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ are called the sides of the triangle.

Axiom 5. If a line cuts one side of a triangle, then it cuts one and only one other side of the triangle unless it goes through a vertex. (Pasch's axiom.)

Def. A set of points is called convex if the set contains the line segment joining any two points of the set.

Theorem 3. For any line k, there exists two convex sets Σ_1 and Σ_2 such that the whole space is the union of k, Σ_1, and Σ_2. Also no two of the sets k, Σ_1 and Σ_2 have a point in common.

Suppose that n is a line and that A and B are points not in n.

Def. A and B are said to be on opposite sides of n if the line segment $\overline{AB}$ intersects n.

Def. A and B are said to be on the same side of n if $\overline{AB}$ does not intersect n.

Problems .

14. Suppose that A,B,C are three points not in the line n. Prove each of the following statements by using the axioms (in particular, axiom 5):

a) If A and B are on the same side of n and if B and C are also on the same side of n, then A and C are on the same side of n.

b) If A and B are on the same side of n and if B and C are on opposite sides of n, then A and C are on opposite sides of n.

c) If A and B are on opposite sides of n and if B and C are on opposite sides of n, then A and C are on the same side of n.

15. Prove that if A and B are on opposite sides of n, then any point C not on n is either on the same side of n as the point A or as the point B.

16. Let A be a given point not in n. We call all points on the same side of n as the point A 'even' points and assign to each such point the number zero. We call points on the opposite side to the point A 'odd' points and assign to each such point the number one. a) Does it make sense to say that two points E and F are on the same side of n if, and only if, the sum of the numbers assigned to them is even? Explain. b) Write each of the statements 14 a), b), c) as a statement about even and odd numbers. c) Could you have used the known facts about even and odd numbers to prove the statements in Problem 14 without using Axiom 5? Explain carefully. (Hint. The answer is 'no'.)

17. Prove Theorem 3.

Def. A <u>map</u> μ of a point set Σ into a point set Σ' is a correspondence or rule which assigns to each point P in Σ exactly one point P' in Σ'. The point P' is called the <u>image</u> of P under the map and we write $P' = \mu(P)$.

Def. A map from Σ to Σ' is said to be <u>distance-preserving</u> if the distance between any two points in Σ equals the distance between their images in Σ'.

Def. The set of points which are on the same side of a line n is called a <u>half-space</u> associated with n.

Lemma 4. Every line n has exactly two half spaces Σ_1 and Σ_2 associated with it. The half-spaces Σ_1 and Σ_2 have no point in common.

Def. A <u>folding</u> φ about the line n is a distance-preserving map of the union of n with one half-space into the union of n with the other half-space such that

a) every point in the second half-space is the image of some point in the first half-space

b) $\varphi(P) = P$ for every point P in n.

Lemma 5. Suppose M is in Σ_1 and M' is the image of M under a folding about n.

a) If P is a point in n, then $d(M,P) = d(M',P)$.

b) If P is in Σ_1, then $d(M',P) > d(M,P)$.

c) If P is in Σ_2, then $d(M,P) > d(M',P)$.

Lemma 6. Suppose that M and M' are the same points as in Lemma 5 and suppose that $\overline{MM'}$ intersects n in the point P. Then $d(M,Q) > d(M,P)$ if Q is a point in n different from P and $d(M,P) = \frac{1}{2}d(M,M')$.

Def. The point P in n which is nearest to a point M not in n is called the <u>orthogonal projection</u> of M onto n.

Axiom 6. For every line n there is at least one folding about n.

Problems

18. Prove Lemma 4.

19. Prove Lemma 5.

20. Suppose that M' is the image of M under a folding about n. Prove each of the following statements:

a) If P is a point such that $d(P,M) = d(P,M')$ then P is in n.

b) If $d(P,M) > d(P,M')$, then P is on the same side of n as the point M'.

c) If $d(P,M') > d(P,M)$, then P is on the same side of n as M.

21. Use Axiom 6 to prove that, for a point M not in n, the orthogonal projection of M onto n exists and is unique.

22. Suppose that φ_1 and φ_2 are foldings about n which map M into M' and M'', respectively. Show that M' = M''. (Hint. If P is the orthogonal projection of M onto n, use the line through M and P, Lemma 5a) and Axiom 2.)

23. Suppose that φ is a folding about n which maps any point P in Σ_1 into its image P' = φ(P) in Σ_2. Consider the map φ^{-1} from $n \cup \Sigma_2$ into $n \cup \Sigma_1$, defined as follows:

i) If Q is in n, then $\varphi^{-1}(Q) = Q$

ii) If Q is in Σ_2, then Q is the image of a point R in Σ_1 and we put $\varphi^{-1}(Q) = R$

a) Prove that $\varphi^{-1}[\varphi(P)] = P$ for any point P in $n \cup \Sigma_1$, and that $\varphi[\varphi^{-1}(Q)] = Q$ for any point Q in $n \cup \Sigma_2$. b) Prove that φ^{-1} is a distance-preserving map. c) Prove that φ^{-1} is a folding about n.

Lemma 7. If Σ_1 and Σ_2 are the half-spaces associated with n, there is exactly one folding about n which maps Σ_1 into Σ_2 and exactly one folding which maps Σ_2 into Σ_1.

Let φ be the unique folding about n which maps Σ_1 into Σ_2.

Def. A reflection about the line n is the mapping ψ of the whole space into itself which is defined as follows:

i) ψ (M) = φ(M) if M is in Σ_1.

ii) ψ (M) = φ^{-1}(M) if M is in Σ_2.

Theorem 4. A reflection about n is a distance-preserving map of the whole space into itself which leaves the points of n fixed.

Problems

24. Prove that a reflection maps the line containing the points P and Q into a line containing the image points P' and Q'.

25. Suppose that M' and Q' are the images of M and Q, respectively, under reflection in n and suppose that the line through M and Q intersects n only in the point P. Prove that the line through M' amd Q' also intersects n in the point P.

26. If M' is the image of M under reflection about n and you know $d(M,Q)$ and $d(M',Q)$ how can you decide whether M and Q are on the same or opposite sides of n, or whether Q is on n?

Def. Two points A and B are symmetrical with respect to a line n if B is the image of A under reflection in n.

Def. A point set Σ is invariant under reflection about a line n if Σ is identical with its image under reflection about n.

Def. The line n is an axis of symmetry for the point set Σ if Σ is invariant under reflection about n.

Problems

27. What kind of triangles have axes of symmetry? What kind of quadrilaterals? What are the axes of symmetry of a circle, of an ellipse, of a plane section of a football through its longest axis?

28. Given a line n. Which configuration of 2 points have n as an axis of symmetry? Which configuration of 3 points, of 4 points, of 5 points?

29. Suppose that the point set Σ is invariant under reflection about n. If Σ contains the points A and B and the line segment $\overline{CD}$, what else must it contain? Will you change your answer if you are told that C and D are on opposite sides of n?

Def. The line n' is perpendicular to n (we write $n' \perp n$) if $n' \neq n$ and if n' is invariant under reflection about n.

Lemma 8. If P is not in the line n and if P' is the image of P under reflection about n, then PP' is perpendicular to n.

Lemma 9. If P is not in the line n, there exists one and only one line which contains P and which is $\perp$ n.

Lemma 10. If the line n' is perpendicular to the line n, then the lines intersect.

Lemma 11. If n' is perpendicular to n, then n is perpendicular to n'.

Problems

30. Prove Lemma 8.

31. Prove Lemma 9.

32. Prove Lemma 10.

Def. A line perpendicular to a segment at the midpoint of the segment is called the <u>perpendicular</u> <u>bisector</u> of the segment.

Lemma 12. If n is the perpendicular bisector of the segment $\overline{AB}$, then A and B are symmetrical with respect to n.

Lemma 13. If d(A,B) = d (A,C) in triangle ABC, then the line through A perpendicular to $\overline{BC}$ is an axis of symmetry for the triangle.

Lemma 14. If each of the points P and Q is equidistant from the points A and B, then A and B are symmetrical with respect to the line through P and Q.

Problems

33. Prove Lemma 11.

34. Prove Lemma 12.

35. Prove Lemma 13.

36. If $d(A,B) = d(A,C)$ in triangle ABC, prove that the line from A to the midpoint of the segment $\overline{BC}$ is perpendicular to the segment $\overline{BC}$.

37. Suppose that the lines n_1 and n_2 are perpendicular to each other. Suppose that A is a point not in either line. Let A_1 be the image of A under reflection in n_1 and let A_2 and A_{12} be the image of the points A and A_1 , respectively, under reflection in n_2. Prove that A_2 and A_{12} are symmetrical with respect to n_1.

Def. A <u>reflection in a point O</u> is a map ξ of the whole space onto itself such that $\xi(O) = O$ and if $P \neq O$, $\xi(P)$ is a point P' on the line through O and P with $d(O, P) = d(O, P')$.

Lemma 15. If the map is repeated twice, each point is mapped onto itself.

Lemma 16. Every line through O is invariant under reflection in O.

Theorem 5. A reflection in a point O is identical to a composition of reflections in each of two arbitrary perpendicular lines intersecting in O.

Notes

1. The image of a point P under reflection in a line n is the point P' which is on a line through P perpendicular to n and which is the same distance from n as the point P is. In other words, the line n is the perpendicular bisector of the line segment $\overline{PP'}$. The line n is called the axis of reflection.

1'. The image of a point P under reflection in a point C is the point P' which is on the line through P and C and which is the same distance from C that P is. In other words, C is the midpoint of the line segment $\overline{PP'}$. The point C is called the center of reflection.

2. Reflection preserves distance and angle.

3. The image of the image of a point is the original point.

4. A line and its image with respect to an axis are either parallel to the axis or they meet on the axis.

4'. A line and its image with respect to a center are either parallel and the center is halfway between the lines or the lines are identical and they pass through the center.

5. A line (point) is an axis (center) of symmetry for a figure if the figure contains the image of each of its points under reflection in the axis (center).

6. If points A and B are each equidistant from the points P and P', then the line AB is an axis of symmetry for the line segment $\overline{PP'}$.

The set of transformations which preserve a given property form a group.

The Euclidean group is the group of all distance preserving transformations. It contains reflections, rotations, translations, and glide reflections. The proper Euclidean group is the subgroup of the Euclidean group which preserves the orientation of figures.

Euc .dean geometry is the study of those properties of figures which are invariant under the transformations of the Euclidean group.

Similarity transformations are those transformations which keep one point fixed and stretch all distances from that point in a fixed ratio. The similarity group is the group generated by similarity transformations and distance preserving transformations.

Affine transformations are transformations which keep two intersecting lines fixed and which stretch lengths parallel to one of the fixed lines by a constant ratio. The affine group is the group generated by the affine transformations and the distance preserving transformations.

A perspective transformation of a plane is obtained by projecting the image of this plane from a given point light source on another plane. The projective group is the group generated by perspective transformations and distance preserving transformations. Projective geometry is the study of those properties of figures which are invariant under the transformations of the projective group.